Linear systems on surfaces with "assigned base points"

X surface, D divisor on X

$|D| = \mathbb{P}(H^0(\mathcal{O}_X(D)))$ linear system.

linear system with assigned base points: $p_1, \dots, p_r$

$$|D - p_1 - \dots - p_r| := \mathbb{P}(\{ s \in H^0(\mathcal{O}_X(D)) \mid \forall i : s(p_i) \})$$

~~A point p_2 is infiniten~~

An infinitely near point p_2 to a point p_1 is a point on the exceptional divisor of the blow up at p_1: $p_2 \in E_{p_1}(X) \subseteq Bl_{p_1} X$
→ indicates tangent direction.

__Exercise__: $|D - p_1 - \dots - p_r| = H^0(Bl_{p_1,\dots,p_r}(X), \mathcal{O}(\beta^* D - E_1 - \dots - E_r))$

__Rmk__ $|D|$ is very ample if:

a) $|D|$ has no base pts

b) $\forall p \in X$ $|D - p|$ has no base pts apart from p (including infinitely near base pts).

encodes separating pts and tangen vectors

~~i.e. iff $\beta_p^* \mathcal{O}_X(D)$ globally generated for all $p \in X$.~~

__Prop__ Consider $\delta_r = |\mathcal{O}_{\mathbb{P}^2}(2) - p_1 - \dots - p_r|$ linear system of conics passing through $p_1, \dots, p_r \in \mathbb{P}^2$.
Suppose no 3 p_i are colinear, $r \leq 4$.
Then δ has no unassigned base pts.

also if p_2 inf near p_1.

__Proof__ δ contains

$$D = \overline{p_1 p_2} + \overline{p_3 p_4} \quad \text{and} \quad D' = \overline{p_1 p_3} + \overline{p_2 p_4}$$

no 3 colinear $\Rightarrow$ $D \cap D' = p_1 + p_2 + p_3 + p_4$. $\square$

if p_2 inf near p_1 then $\overline{p_1 p_2}$ = line determined by tangent dir. of p_2

__Cor__ $r \leq 5$

a) $\dim \delta_r = \binom{2+2}{2} - 1 - r = 5 - r$

b) $\exists!_1$ conic passing through $p_1, \dots, p_5$, irreducible.

also if p_5 inf. near $p_1, \dots, p_4$

__Pf__ a) each base pt drops dim by 1, b) irred. b/c none colinear.

Proposition $\delta_r^3 = |\mathcal{O}_{\mathbb{P}^2}(3) - p_1 - \dots - p_r|$ linear system of plane cubic curves passing through $p_1, \dots, p_r$.

Suppose i) no 4 of the p_i are collinear and
ii) no 7 of them lie on a conic. remains true if p_2 inf. close to p_1

$r \leq 7$

Then δ_r^3 has no unassigned base points.

Proof $p_1, \dots, p_7$ ord. points. $q \in \mathbb{P}^2 \setminus \{p_1, \dots, p_r\}$.

Show: $\exists$ cubic passing through $p_1, \dots, p_7$ but not q.

"special line" Case 1: $p_1, p_2, p_3 \in L \ni q$

p_4, p_5, p_6, p_7 not all colinear wlog p_4, p_5, p_6 not collinear.

$\Rightarrow \Gamma_{12456}$ conic through $p_1 p_2 p_4 p_5 p_6$

$\Rightarrow C = \Gamma_{12456} + \overline{p_3 p_7}$ cubic curve containing $p_1, \dots, p_7$

if $q \in \Gamma_{12456}$, then $L \subset \Gamma_{12456}$ $\Rightarrow \Gamma$ reducible.

$\Rightarrow p_4, p_5, p_6$ colinear. ↯

if $q \in \overline{p_3 p_7} \Rightarrow p_7 \in L \Rightarrow p_1, p_2, p_3, p_7$ colinear ↯

"special conic" Case 2: q not colinear with any set of three points

but $q \in \Gamma \ni p_1, \dots, p_6$.

$\Rightarrow \Gamma_{12347} + L_{56}$ doesn't contain q.

$L_{56} = \overline{p_5 p_6}$

"generic" Case 3: else.

One of the cubic curves

$\Gamma_{1234i} + L_{jk}$ $\quad i,j,k = (5,6,7), (6,5,7), (7,6,5)$

does not contain q.

Exercise: details ....

□

Theorem δ^3_r as before
Suppose (i) no 3 p_i colinear
(ii) no 6 lie on a conic. $\}$ generic

If $r \leq 6$, then $\beta^* \delta^3_r$ on $Bl_{p_1,\dots,p_r} \mathbb{P}^2$ is very ample.

Proof Apply proposition to check that $\beta^*\delta^3_r$ separates points and tangents! $\square$

Corollary $p_1,\dots,p_6$ generic, $X = Bl_{p_1,\dots,p_6} \mathbb{P}^2 \hookrightarrow \mathbb{P}\{\beta^* \delta^3_6\}^* \cong \mathbb{P}^3$
is a cubic surface, $\omega_X = \mathcal{O}_{\mathbb{P}^3}(-1)$

Proof $\dim(\beta^*\delta^3_6) = \dim(\delta^3_6) = \underbrace{\binom{2+3}{3}}_{=10} - 1 - 6 = 3$

$D \in \beta^*\delta^3_6$

$D^2 = (\beta^*\mathcal{O}(3) - E_1 - \dots - E_6)^2 = 9 - 6 = 3$

$\Rightarrow \deg_{\mathbb{P}^3}(X) = 3$

$\omega_X = \mathcal{O}_{\mathbb{P}^3}(-1)$ by adjunction formula or by formula for canonical divisors on blow ups.

$$\omega_X = \beta^* \omega_{\mathbb{P}^2} + E_1 + \dots + E_6 = -D \qquad \square$$

Prop $X = Bl_{p_1,\dots,p_6} \mathbb{P}^2$, $p_1,\dots,p_6$ generic.

a) $\mathrm{Pic}(X) \cong \mathbb{Z}^7 = \mathbb{Z}\beta^*\mathcal{O}(1) \oplus \mathbb{Z}E_1 \oplus \dots \oplus \mathbb{Z}E_6$

b) intersection matrix $\begin{pmatrix} 1 & & & & & & 0 \\ & -1 & & & & & \\ & & -1 & & & & \\ & & & -1 & & & \\ & & & & -1 & & \\ & & & & & -1 & \\ 0 & & & & & & -1 \end{pmatrix}$

c) $h = \mathcal{O}_{\mathbb{P}^3}(1)|_X = 3l - \sum e_i$

d) $K = -h = -3l + \sum e_i$

$D \subset X$ effective e) $\deg_{\mathbb{P}^3}(D) = 3a - \sum b_i$
$\| \; al - \sum b_i e_i$

f) $D^2 = a^2 - \sum b_i^2$

g) $p_a(D) = \frac{1}{2}(D^2 - d) + 1 = \frac{1}{2}(a-1)(a-2) - \frac{1}{2}\sum b_i(b_i - 1)$

Theorem (27 lines)

$X = Bl_{P_1,\dots,P_6}\mathbb{P}^2$ $\quad P_i$ generic.

X contains exactly 27 lines, all are -1 curves!

(i) the exceptional divisors $E_1,\dots,E_6$ $\quad 6$

(ii) $\hat{L}_{ij}$ strict transforms of line $L_{ij} = \overline{P_iP_j}$ $\quad 15$

(iii) $\hat{\Gamma}_j$ strict transform of conics through P_i, $i \neq j$. $\quad 6$

Proof h hyperplane class, K canonical class

$\Rightarrow h \equiv_{num} -K$ $\qquad C_0 \subset X$ genus zero, $\deg_{\mathbb{P}^3}(C_0) = 1$

$$\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad (h \cdot C_0)$$

genus formula $\Rightarrow -2 = (C_0 \cdot C_0 + K)$

$\Rightarrow C_0^2 = -1$

$\Rightarrow C_0$ -1-curve.

Suppose $C_0 \neq E_1,\dots,E_6$

$\Rightarrow (C_0 \cdot E_i) = 0$ or 1 $\qquad$ distinct lines intersect w/ mult at most 1. or don't intersect

$\Rightarrow C_0 \sim_{num} m\,\beta^*\mathcal{O}_{\mathbb{P}^2}(1) - \sum (C_0 \cdot E_i) E_i$

$\Rightarrow 1 = (C_0 \cdot H) = 3m - \sum m_i$

$\Rightarrow (m, m_i) = \begin{cases} (1, \text{ 2 of } m_i = 1) & \to \text{strict transform of lines} \\ (2, \text{ 5 of } m_i = 1) & \to \text{strict transform of } \Gamma_j \end{cases}$

Exercise to check that the strict transform satisfy these descriptions. $\square$

Theorem Every nonsingular cubic surface is ~~the~~ a blow up of 6 distinct points on $\mathbb{P}^2$

Proof (implies 27 lines $\forall$ cubics)

Step 1 Every cubic surface contains a line.

$$\begin{tikzcd} & \mathbb{P}^{19} \times \mathbb{G}(1,3) \supset Z \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathbb{P}^{19} \cong \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(3))) & & \mathbb{G}(1,3) \cong Gr(2,4) \subset \mathbb{P}^5 \end{tikzcd}$$

moduli of cubics in $\mathbb{P}^3$

dim 4

moduli of lines ℓ in $\mathbb{P}^3$: determined by $\lambda_1, \lambda_2 \in H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ linearly independent

$H^0(\mathcal{O}_{\mathbb{P}^3}(1)) = \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}z \oplus \mathbb{C}w$

$p^{-1}(V(z,w))$

$= \{$ cubics defined by equations $f(x,y,z,w)$ with coeff of monomials $x^3, x^2y, xy^2, y^3 = 0 \}$

$= \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(3)) / \mathrm{span}\{x^3, x^2y, xy^2, y^3\}) = \mathbb{P}^{15}$

$\Rightarrow \dim(Z) = 19$

if q not surjective, ~~then codim q(Z)~~

$\Rightarrow \operatorname{codim} q(Z) \geq 1$

$\Rightarrow q^{-1}(V(f)) = \left\{ \begin{matrix} \emptyset \\ \text{or} \\ \text{positive dim'l} \end{matrix} \right\} \forall f.$

but $\exists V(f)$ w/ finitely many lines $\lightning$

$\overset{\shortparallel}{Bl_{p_1,\dots,p_6}\mathbb{P}^2}$

$\Rightarrow q$ surjective

$\rightarrow$

Step 2 X ns cubic surface, $l \subset X$ line

- Then there are exactly 10 lines $l_1, l_1', \dots, l_5, l_5'$ meeting l, which come in pairs that intersect.
- X contains two disjoint lines.

Consider the pencil (= 1-dim lin. system) of hyperplanes containing l: $\{P_t\}_{t \in \mathbb{P}^1}$

$P_t \supset \underset{\text{cubic}}{X \cap P_t} = l \cup C_t$, $C_t \subset P_t$ conic.

C_t doesn't contain l or a double line.

Suppose $P_t \cap X = V(\lambda_t^2) \cup V(\mu_t)$

$\Rightarrow X = V(\gamma_t q_t + \lambda_t^2 \mu_t)$ $\quad q_t \in H^0(\mathcal{O}_{\mathbb{P}^3}(2))$, $P = V(\gamma_t)$

$d(\gamma_t q_t + \lambda_t^2 \mu_t) = q_t d\gamma + \gamma dq + 2\lambda\mu d\lambda + \lambda^2 \mu_t d\mu$.

vanishes on $V_+(q_t, \gamma_t, \lambda_t) \underset{\deg 2}{\subseteq} \underset{\text{line}}{V(\gamma_t, \lambda_t)}$

$\Rightarrow X$ singular ↯

$\Rightarrow C_t$ irred. ns. conic or C_t union of distinct intersecting lines. (~~for~~ coplanar b/c)

$\mathbb{P}^3_{x,y,z,w}$ write $l = V(z,w)$

$$\Rightarrow X = V(\overbrace{a_1 x^2 + 2b_1 xy + c_1 y^2 + 2d_2 x + 2e_2 y + f_3}^{=:F})$$

$a_1, b_1, c_1, d_2, e_2, f_3 \in k[z,w]$, homogeneous of indicated degree

$C_t \subseteq P_t$ cut out by $\dfrac{F(x,y,tw,w)}{w}$ quadric w/ coeff $a - f$

$P_t = V(z - tw)$

$\Rightarrow C_t$ singular when (i.e. quad. form degenerate) $\Delta(\underset{tw}{z}, w) = \det \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} = 0$

$= a_1 c_1 f_3 + 2 b_1 e_2 d_2 + c_1 d_2^2 + a_1 e_2^2 + b_1^2 f_3$

quintic equation — skip details (key point)

smoothness of $X \rightsquigarrow$ no repeated roots. $\longrightarrow$

Step 3 Construction of birational map.

$l, l' \subset X$ two disjoint lines.

$\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ projection from l $(\mathcal{O}(1), z, w)$, $V(\cancel{z}, z, w)$

$\pi' : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ — " — l'

π defined on all of X:

[base locus of π $l : V(z,w) \subset \mathbb{P}^3$, resolve π by blowing up l: $\tilde{\pi} : Bl_l \mathbb{P}^3 \to \mathbb{P}^1$
on X $Bl_l X = X$ b/c $l \subset X$ divisor.

$\Rightarrow$ get morphism $\psi : X \xrightarrow{\pi \times \pi'} \mathbb{P}^1 \times \mathbb{P}^1 = l' \times l$.

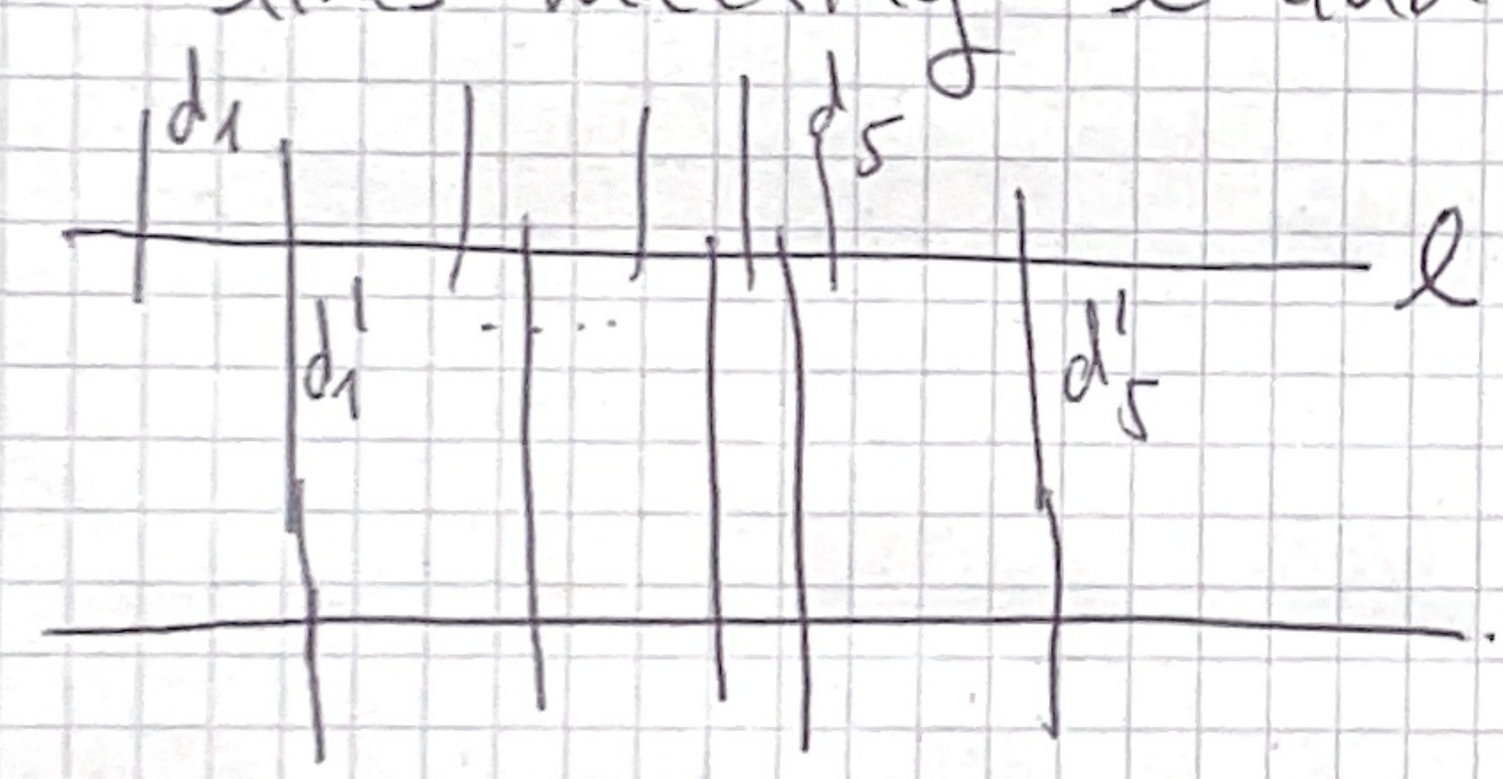
$l' \times l \overset{\varphi}{\dashrightarrow} X$

$(p,q) \longmapsto \underbrace{(\overline{pq} \cap X)}_{\substack{3 \text{ points} \\ (\text{generally})}} - \{p,q\}$.

Step 4 $\Rightarrow \psi : X \xrightarrow{\pi \times \pi'} \mathbb{P}^1 \times \mathbb{P}^1$ birational.

$\Rightarrow$ sequence of blow ups. determined by contracted curves

= lines meeting l and l'



$X \cap P_i = l \cup d_i \cup d_i' \subseteq P_i$ plane. must intersect $l' \cap P_i \in \cancel{l} \cup d_i \cup d_i'$ only one else coplanar.

$\Rightarrow \psi$ contracts 5 lines

$\Rightarrow X = Bl_{p_1, \dots, p_5}(\mathbb{P}^1 \times \mathbb{P}^1)$

$\cong Bl_{p_1, \dots, p_4}(Bl_{p_5}(\mathbb{P}^1 \times \mathbb{P}^1))$

$\cong Bl_{p_1, \dots, p_4}(Bl_{p_5, p_6}(\mathbb{P}^2)) \cong Bl_{p_1, \dots, p_6}(\mathbb{P}^2)$ $\square$